A Survey of Applications of Fuzzy Orderings: From Databases to Statistics and Machine Learning

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Abstract:
This contribution provides an overview of practical applications of similarity-based fuzzy orderings. These applications include flexible database querying, robust statistics, natural language semantics, and fuzzy rule-based machine learning.

Keywords:
fuzzy ordering, fuzzy relation, machine learning.

1 Introduction

It is an outstanding feature of human thinking to inherently employ a gradual concept of equality with a tolerance for imprecision. Consider the notorious example how to define a set of tall people. Specifying a sharp limit, e.g. 180cm, leads to unnatural preciseness. While a person of 179.95cm would be classified as not tall, somebody of 180.05cm would be classified as tall, although it is virtually impossible to distinguish between the two. Allowing a gradual transition of membership between the two classes "not tall" and "tall"—as fuzzy sets do—solves this problem in a simple and pragmatic way. This viewpoint suggests that gradual similarity is, in some sense, an inherent component of fuzziness. The example also indicates that this kind of gradual similarity appears in ordinal contexts too, as the height of people and the concept "tall" have an ordinal structure.

Equivalence relations and orderings are Siamese twins in classical mathematics. Within the early gold rush of fuzzification of any classical mathematical concept, these two fundamental types of relations did not have to await the introduction of their fuzzy counterparts for a long time [35].

On the one hand, a rich theory of *fuzzy equivalence relations* has been developed since the 1970ies. It has turned out that fuzzy equivalence relations are most suitable concepts for modeling gradual similarity in the sense discussed above. The study of fuzzy equivalence relations has not only brought insight into this class of fuzzy relations, but also deep insight into the inherent principles underlying fuzzy sets [13, 29, 30, 34].

On the other hand, different concepts of *fuzzy orderings* have existed, but the theory remained underdeveloped. This is astonishing and seems almost paradoxical, as almost all fuzzy systems make implicit use of ordinal structures—there might be only a small minority of fuzzy systems in which expressions, such as "small", "medium", or "large', do not occur. This was the starting point of a long-term research effort the outcome of which is subsumed in the present contribution. The long-term goal was not only to create a reasonable theory of fuzzy orderings, but also to achieve the integration of this concept into real-world applications. This talk aims at providing an overview from the viewpoint of applications.

2 "Similarity-Based" Fuzzy Orderings

In classical mathematics, equivalence relations and orderings are both special cases of preorderings (reflexive and transitive relations). While equivalence relations can be constructed as symmetric kernels of preorderings, orderings are obtained from preorderings by factorization with respect to this symmetric kernel. This fundamental correspondence did not hold for fuzzy orderings in the sense of Zadeh [35]. The relatively unknown paper [26], for the first time, proposes a generalization that explicitly links fuzzy orderings to fuzzy equivalence relations in a way analogous to classical mathematics. This idea has been revitalized and further developed in [1–3].

Definition 1 A binary fuzzy relation L on a domain X is called *fuzzy ordering* with respect to a t-norm T and a T-equivalence E on X, for brevity *T-E-ordering*, if it fulfills the following three axioms for all $x, y \in X$:

(i) E-Reflexivity: $E(x,y) \leq L(x,y)$

(ii) T-E-antisymmetry: $T(L(x,y), L(y,x)) \leq E(x,y)$

(iii) T-transitivity: $T(L(x,y), L(y,z)) \leq L(x,z)$

It is not the main focus of this paper to go deeply into constructions and representations of fuzzy orderings. We only mention an important subclass that will be of particular practical interest throughout the remaining paper.

Definition 2 A T-E-ordering L is called *strongly linear* if, for all $x, y \in X$,

$$\max(L(x,y), L(y,x)) = 1.$$

As the term "strong linearity" suggests, this property can be considered as a generalization of the classical linearity property. However, strong linearity is usually too strong a requirement to be acceptable as a general concept of linearity [10].

Definition 3 A crisp ordering $\preceq$ on a domain X and a T-equivalence E on X are called *compatible*, if and only if the following holds for all $x, y, z \in X$:

$$x \preceq y \preceq z \Rightarrow E(x,z) \leq \min(E(x,y), E(y,z))$$

Compatibility between a crisp ordering $\preceq$ and a fuzzy equivalence relation E can be interpreted as follows: the two outer elements of an ordered three-element chain are at most as similar as any two inner elements.

Theorem 3.1 [1,2] *Consider a binary fuzzy relation L on a domain X and a T-equivalence E on X. Then the following two statements are equivalent:*

(i) L is a strongly linear T-E-ordering.

(ii) There exists a linear ordering $\preceq$ the relation E is compatible with such that L can be represented as follows:

$$L(x,y) = \begin{cases} 1 & \text{if } x \preceq y \\ E(x,y) & \text{otherwise} \end{cases} \quad (1)$$

Theorem 3.1 allows to consider strongly linear T-E-orderings as "linear orderings with imprecision", which are common phenomena in everyday life. Reconsider the example of comparing the heights of people. Although we have a clear crisp concept for ordering heights (which are just positive real numbers), there is undoubtedly a certain tolerance for imprecision or indistinguishability in the way we actually perform such a comparison. Similar situations occur in virtually any application where values have to be processed for which a crisp ordering would exist, but where the distinction of small differences is either impossible or unnecessary.

3 Flexible Query Answering Systems

The use of fuzzy equivalence relations has had a long tradition in flexible query answering systems (FQAS's) [23, 27, 31, 32], while fuzzy orderings have never been applied in this domain. The reason was simply that previous approaches to fuzzy orderings were not rich enough to fulfill the needs of applications in FQAS's. In the new framework, linear orderings with imprecision (see Theorem 3.1 and the discussion thereafter) provide a simple and elegant means to interpreting ordinal queries in a flexible manner.

In [12], we have proposed a way how to further enrich flexible querying by fuzzy orderings. This has been demonstrated by a prototype system that acts as a proxy to an SQL database. This system uses an extension of SQL in which the conditions “IS”, “IS AT LEAST”, “IS AT MOST”, and “IS WITHIN” can be interpreted in a fuzzy way (generalizing the standard SQL constructs “=”, “>=”, ‘<=”, and “BETWEEN”, respectively). Provided that we are given an attribute on a domain X for which we know a crisp linear ordering $\preceq$ and a T-equivalence E which is compatible to $\preceq$, the fuzzy relation defined in (1) is a strongly linear T-E-ordering. Then we can compute the degrees of fulfillment of the following query fragments in the following way (for a query value q and a record x):

$$\begin{aligned} t(\text{“}x \text{ IS } q\text{”}) &= E(q,x) \\ t(\text{“}x \text{ IS AT LEAST } q\text{”}) &= L(q,x) \\ t(\text{“}x \text{ IS AT MOST } q\text{”}) &= L(x,q) \\ t(\text{“}x \text{ IS WITHIN } (a,b)\text{”}) &= \\ & T\big(L(\min(a,b),x), L(x,\max(a,b))\big) \end{aligned}$$

The standard case is $X = \mathbb{R}$ (or $X \subset \mathbb{R}$). In this case, we can use simple distance-based constructions to define E and L in an intuitive and interpretable way [7, 12, 17].

4 Robust Rank Correlation Measures

Rank correlation measures are intended to measure to which extent a monotonic function is able to model the dependence between the two observables. They neither assume a specific parametric model nor specific distributions of the observables. They can be applied to ordinal data and, if some ordering relation is given, to numerical data too. Therefore, rank correlation measures are ideally suited for detecting monotonic relationships, in particular, if more specific information about the data is not available. The most common approaches are *Spearman's rho* [33], *Kendall's tau* [28], and *Goodman's and Kruskal's gamma* [24].

All these rank correlation measures need to correct for ties, i.e. equally ranked observations. For real-valued data, these corrections fail, since they are based on crisp equalities. In particular in the presence of noise, it would be desirable to handle ties such that the extent to which monotonicity is violated is taken into account too. This can be achieved by using fuzzy orderings in a straightforward way. In [11], we have proposed a fuzzy ordering-based extension of Goodman's and Kruskal's gamma.

In the classical setting, for a given set of pairs of observations $(x_i, y_i)_{i=1}^n$, the numbers of concordant and discordant pairs are defined as follows:

$$\begin{aligned} C &= |\{(i,j) \mid x_i < x_j \text{ and } y_i < y_j\}| \\ D &= |\{(i,j) \mid x_i < x_j \text{ and } y_i > y_j\}| \end{aligned}$$

The classical gamma measure is then defined as

$$\gamma = \frac{C-D}{C+D}.$$

It is obvious that the above definitions are highly sensitive to small random variations of the data. Our idea is based on replacing the crisp strict ordering $<$ by a strict fuzzy ordering.

Given a T-E-ordering L, the fuzzy relation

$$R(x,y) = \min(L(x,y), N_T(L(y,x))),$$

where $N_T(x) = \sup\{y \in [0,1] \mid T(x,y) = 0\}$ is the residual negation of T, is the most appropriate choice for extracting a strict fuzzy ordering from a given fuzzy ordering L (for a detailed argumentation, see [9]).

If we assume that R_X is an appropriate strict fuzzy ordering for the first component (the x values) and R_Y is a strict fuzzy ordering for the second component (the y values), we can compute the degree to which (i,j) is a concordant pair as

$$\tilde{C}(i,j) = \bar{T}(R_X(x_i,x_j), R_Y(y_i,y_j))$$

and the degree to which (i,j) is a discordant pair as

$$\tilde{D}(i,j) = \bar{T}(R_X(x_i,x_j), R_Y(y_j,y_i)),$$

where $\bar{T}$ is some t-norm to aggregate the relationships of x and y components. If we adopt the simple sigma count idea to measure the cardinality of a fuzzy set [18], we can compute the numbers of concordant pairs $\tilde{C}$ and discordant pairs $\tilde{D}$, respectively, as

$$\tilde{C} = \sum_{i=1}^{n} \sum_{j \neq i} \tilde{C}(i, j), \quad \tilde{D} = \sum_{i=1}^{n} \sum_{j \neq i} \tilde{D}(i, j).$$

Finally, we can define a *fuzzy ordering-based rank correlation measure* $\tilde{\gamma}$ as

$$\tilde{\gamma} = \frac{\tilde{C} - \tilde{D}}{\tilde{C} + \tilde{D}}.$$

It is worth to point out that, in case that the strict fuzzy orderings and the operations involved are continuous, $\tilde{\gamma}$ depends continuously on the input data $(x_i, y_i)_{i=1}^{n}$. Further analyses [11] suggest that this new way of measuring correlations is highly suitable for noisy data—although the analyses are still far from being statistically profound.

5 Ordering-Based Modifiers

Ordering-based modifiers, such as *'at least'*, *'at most'*, or *'between'* are ubiquitous in natural language. The queries highlighted in Section 3 above also make use of these constructs. Given a fuzzy ordering L on the domain under consideration, we can use the same way as in Section 3 to define fuzzy sets modeling the expressions *'at least q'*, *'at most q'*, or *'between r and q'*, where r and q are crisp values. From that point of view, fuzzy orderings also provide us with means to model ordinal predicates that we can use directly in fuzzy systems applications.

For many fuzzy systems, however, it might be desirable to have a way of defining the meaning of expressions like *'at least medium'*, where *'medium'* is a fuzzy set. It is clear that it would be most convenient if we could define the semantics of the modifier *'at least'* in a strictly functional manner, i.e. that the fuzzy set modeling *'at least A'* is a function of the fuzzy set modeling the expression *'A'*.

If we are given a T-E-ordering L on the domain under consideration X, we can use direct images with respect to fuzzy relations [25] to achieve this goal (for brevity, ATL stands for *'at least'* and ATM stands for *'at most'*) [8]:

$$\begin{aligned} &\mathrm{ATL}_L(A)(x) \\ &\qquad = \sup\{T(A(y), L(y, x)) \mid y \in X\} \\ &\mathrm{ATM}_L(A)(x) \\ &\qquad = \sup\{T(A(y), L(x, y)) \mid y \in X\} \end{aligned}$$

If a crisp ordering $\preceq$ is considered, the above definitions simplify to

$$\begin{aligned} \mathrm{ATL}_{\preceq}(A)(x) &= \sup\{A(y) \mid y \preceq x\}, \\ \mathrm{ATM}_{\preceq}(A)(x) &= \sup\{A(y) \mid x \preceq y\}. \end{aligned}$$

If L is a strongly linear T-E-ordering that is a direct fuzzification of a crisp linear ordering $\preceq$, we can prove that the equalities [8]

$$\begin{aligned} \mathrm{ATL}_L(A) &= \mathrm{ATL}_{\preceq}(\mathrm{EXT}_E(A)) \\ &= \mathrm{EXT}_E(\mathrm{ATL}_{\preceq}(A)) \\ \mathrm{ATM}_L(A) &= \mathrm{ATM}_{\preceq}(\mathrm{EXT}_E(A)) \\ &= \mathrm{EXT}_E(\mathrm{ATM}_{\preceq}(A)) \end{aligned}$$

hold for all fuzzy sets $A \in \mathcal{F}(X)$, where

$$\begin{aligned} &\mathrm{EXT}_E(A)(x) \\ &\qquad = \sup\{T(A(y), E(y, x)) \mid y \in X\} \end{aligned}$$

is the extensional hull of A with respect to E. This means that, for the practically important case of direct fuzzifications, the image of the fuzzy ordering is obtained from the images of the crisp ordering and the fuzzy equivalence relation (which then even commute). Figure 1 shows an example of this correspondence.

These ordering-based modifiers can be applied directly in fuzzy modeling and, equally important, in fuzzy rule-based machine learning. Ordering-based modifiers can be used to introduce additional linguistic expressions, possibly allowing more expressive and more compact rule systems, *without compromising interpretability in any way*. Indeed, several fuzzy

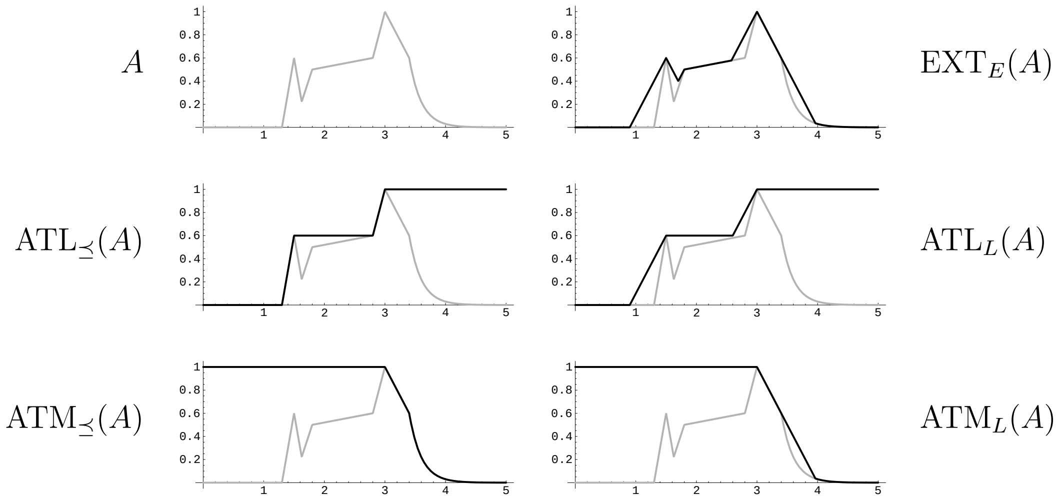

Figure 1: A fuzzy set and the results obtained by applying various ordering-based modifiers.

rule-based machine learning methods have been introduced in recent years that make use of ordering-based modifiers [19–22]. These methods have been used successfully in industrial applications and they are available as part of a commercial machine learning software.[1]

6 Further Topics

Orderings of fuzzy sets. Based on the definition of ordering-based modifiers, it is possible to define a preordering of arbitrary fuzzy sets on any domain for which a T-E-ordering L is known [4, 5]. This relation is defined as

$$A \preceq_L B \Leftrightarrow \big(\mathrm{ATL}_L(A) \supseteq \mathrm{ATL}_L(B) \text{ and } \mathrm{ATM}_L(A) \subseteq \mathrm{ATM}_L(B)\big),$$

where $\subseteq$ denotes the usual crisp inclusion of fuzzy sets. It is easy to see (compare with Fig. 1) that the first inclusion $\mathrm{ATL}_L(A) \supseteq \mathrm{ATL}_L(B)$ corresponds to the fact that the left flank of A is above (to the left) of the left flank of B, while the second inclusion $\mathrm{ATM}_L(A) \subseteq \mathrm{ATM}_L(B)$ means that the right flank of A is below (to the left) of the right flank of B.

The unique feature of this approach is that it can be applied to any kind of fuzzy sets on a domain for which a crisp or fuzzy ordering is known. In particular, no special assumptions concerning the structure of the space X (e.g., linearity of the ordering, restriction to real numbers or intervals, etc.) have to be made. For more details on the properties of the relation $\preceq_L$, see [4, 5].

[1] `http://www.unisoftwareplus.com/products/mlf/`

Interpretability of Linguistic Variables Interpretability is often used a selling argument to promote fuzzy systems. In recent years, the community has increasingly acknowledged the fact that interpretability is not a feature that fuzzy systems have per se. Instead, interpretability has to be ensured by appropriate constraints in order to avoid that a fuzzy system degenerates to a black box [15].

In [6], we have proposed an axiomatic approach to the interpretability of linguistic variables. It is based on the idea that the fuzzy sets modeling the linguistic expressions should also obey the same relationships one would expect from the linguistic expressions. As a simple example, the fuzzy sets modeling *'small'*, *'medium'*, and *'large'* should be in proper order. Since such ordinal expressions are ubiquitous in fuzzy modeling, it is clear that orderings of fuzzy sets are essential for assessing interpretability. The paper [6] elaborates these thoughts until, in the end, practically feasible interpretability constraints are devised.

Using these constraints in fuzzy rule-based machine learning leads to more complex optimization problems that require more sophisticated solution strategies. One option are genetic algorithms and other heuristics [16]. In [14], a regularized numerical algorithm is proposed that allows for a quick and reliable optimization of Sugeno fuzzy systems.

7 Conclusion

This paper is intended as a pleading for the importance of fuzzy orderings in applications aside of preference modeling and decision analysis. In order to support this claim, several fields of practical applications have been discussed. They clearly underline that fuzzy orderings are not just of pure theoretical interest, but can also have fruitful practical applications.

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